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# A SEPARABLE POSTLIMINAL $C^*$-ALGEBRA WITHOUT MAXIMAL CLOSED IDEALS

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**Abstract.** Let $G$ be the free abelian group with a countable number of generators. We construct a separable locally compact $G$-transformation space $X$ without closed minimal invariant subsets, such that the corresponding $C^*$-algebra $C^*(G, X)$ has the properties mentioned in the title. Using $X$ we also give an example of a transformation space $(G, Z)$ without closed minimal invariant subset, on which $G$ acts freely.

The belief that liminal or postliminal $C^*$-algebras behave in some sense almost like finite-dimensional or commutative $C^*$-algebras certainly is not very well founded, see [1, esp. the examples in 4.7]. Here we will construct another exotic example with properties as mentioned in the title. It is probably not surprising that such an algebra exists, see [1, 4.7.17.b], but our construction is based on the theories of transformation spaces, covariant representations and crossed products which were developed during the last few years by Effros and Hahn, Glimm, Guichardet, Takesaki, Zeller-Meier, myself and others (see the bibliography), and also settles a question raised in [2, 7.2, p. 80]. It might be possible to refine our construction and to produce $C^*$-algebras $\mathscr{A}$ with the following property: If $I$ and $J$ are closed two-sided ideals in $\mathscr{A}$ with $I \subsetneq J$, then there exists a closed ideal $M$ with $I \subsetneq M \subsetneq J$. Of course such an algebra has to be antiliminal.

A locally compact *transformation space* is a pair $(G, Z)$ of a locally compact Hausdorff space $Z$, a locally compact group $G$ and a continuous mapping $G \times Z \to Z$ with the usual properties. Let $\mathscr{Z}$ be the Banach algebra of continuous complex-valued functions on $Z$ vanishing at infinity. The action of $G$ on $Z$ implies a strongly continuous homomorphism of $G$ into the group of isometric automorphisms of $\mathscr{Z}$, hence $\mathscr{Z}$ is a $G-C^*$-algebra and we may form the generalized $L^1$-algebra $\mathscr{L}^1(G, \mathscr{Z})$ (with trivial factor-system $P$), see e.g. [6].

Let $\mathscr{C}^*(G, Z)$ be the $C^*$-hull of $\mathscr{L}^1(G, \mathscr{Z})$ and Prim $(G, Z)$ the primitive ideal space of $\mathscr{C}^*(G, Z)$. There is a 1:1-correspondence between the unitary representations $\Pi$ of $\mathscr{C}^*(G, Z)$ and the covariant representations $\pi = (\pi_1, \pi_2)$ of $(G, Z)$. Here $\pi_1$ is a continuous unitary representation of $G$ and $\pi_2$ a unitary representation of $\mathscr{Z}$ in the same Hilbert space such that $\pi_1(g)^*\pi_2(a)\pi_1(g) = \pi_2(a^g)$ for $g \in G$, $a \in \mathscr{Z}$,

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$a \to a^g$ the automorphism induced by $g$ on $\mathscr{Z}$. The hull $h(\ker \pi_2) = H$ is a closed $G$-invariant subset of $Z$. Guichardet has proved in [4] that if $Z$ is separable and $\pi$ is irreducible, then $H$ is the closed hull cl $(Gx)$ of an orbit $Gx = \{gx; g \in G\}$ of an element $x \in Z$. This closure $x^\sim = \mathrm{cl}\,(Gx)$ depends only on $\ker \Pi = p \in \mathrm{Prim}\,(G, Z)$; so there is a natural mapping

$$\alpha\colon \mathrm{Prim}\,(G, Z) \to (Z/G)^\sim$$

into the set $(Z/G)^\sim$ of all orbit closures $x^\sim$, $x \in Z$. The projection $x \to x^\sim$ defines a quotient topology on $(Z/G)^\sim$ with respect to which $\alpha$ is continuous. *If $G$ is discrete, amenable and acts freely on $Z$* (no $g \neq e$ in $G$ has a fixed point), *then $\alpha$ is a homeomorphism* ([2, Corollary 5.16], [9, Théorème 5.15]). *If furthermore all orbits $Gz$ are discrete in $Z$ then $\mathscr{C}^*(G, Z)$ is postliminal* [9, Théorème 7.7].

Let $G$ be the discrete free abelian group with free generators $\tau_n$, $n = 1, 2, \ldots$. At first we will construct a countable locally compact transformation space $(G, X)$ without nonempty closed minimal $G$-invariant subsets and with discrete orbits. Unfortunately $G$ does not act freely on $X$, so we cannot apply Theorem 7.7 from [9] to prove that $\mathscr{C}^*(G, X)$ is postliminal. Fortunately it is easy to see directly that $\mathscr{C}^*(G, X)$ is postliminal without maximal closed ideals.

On the other hand it is even simpler to construct from $X$ a separable locally compact transformation space $(G, Z)$ without minimal closed invariant subspaces and *free* action of $G$, but with nondiscrete orbits. Now [2, Corollary 5.16] tells us that $\mathscr{C}^*(G, Z)$ has no maximal ideals.

*Construction of* $(G, Z)$. For $n = 0, 1, 2, \ldots$ let

$$c_n = \exp\,(\pi i 2^{-n}), \qquad c_{-n} = \bar{c}_n,$$

and let $K$ be the closed hull of the set of all $c_n$, $n \in \boldsymbol{Z}$, i.e.

$$K = \{1, c_0, c_{\pm 1}, c_{\pm 2}, \ldots\}.$$

Then $K$ is a closed subset of the unit circle $\boldsymbol{T}$ in the complex plane with 1 as the only accumulation point. Define a homeomorphism $\tau$ of $K$ by

$$\tau 1 = 1, \qquad \tau c_n = c_{n+1}.$$

We have $\lim_{n \to \infty} \tau^n x = 1$ for every $x \in K$ and $\tau x = x$ if and only if $x = 1$.

Let $X$ be the set of all sequences $x = \{x_i\}_{i \in \boldsymbol{N}}$ of elements $x_i \in K$ such that $x_i \neq 1$ implies $x_{i+1} \neq 1$ and $x_j = x_{j+1} \neq 1$ for almost all $j \in \boldsymbol{N}$:

$$x = \{1, 1, \ldots, 1, x_k, x_{k+1}, \ldots, x_l, x_l, x_l, \ldots\}, \qquad x_j \neq 1 \text{ for } j \geqq k.$$

For $\varepsilon > 0$ let

$$U_\varepsilon(x) = \{y \in X; y_j = x_j \text{ for } j \geqq k, |1 - y_{k-1}| \leqq \varepsilon\},$$

(where $x = \{x_i\}$ with $x_i = 1$, $i < k$, $x_k \neq 1$). We have $x \in U_\varepsilon(x)$ and $U_\eta(x) \subset U_\varepsilon(x)$ if $0 < \eta \leqq \varepsilon$. Let $z = \{z_i\} \in U_\varepsilon(x)$, $z \neq x$. Then necessarily $U_\eta(z) \subset U_\varepsilon(x)$ for all $\eta > 0$

because no condition is imposed on the first $k-2$ components of the elements in $U_\varepsilon(x)$. If $y \neq x$, $y_{k-1} \neq 1$ and $\varepsilon < |1-y_{k-1}|$ then $U_\eta(y) \cap U_\varepsilon(x) = \varnothing$ for every $\eta > 0$. If $y_{k-1}=1$, $y_k \neq 1$, then $U_\varepsilon(x) \cap U_\varepsilon(y) = \varnothing$ for all $\varepsilon > 0$. Taking the $U_\varepsilon(x)$ as a basis for the neighborhoods of $x$ we get a Hausdorff topological space $X$. Because $X$ is countable and every point has a countable basis of open neighborhoods, the space $X$ has a countable basis of open sets, i.e. $X$ is separable. Let $\{x^n\}_n$ be a sequence in $U_\varepsilon(x)$, $x^n = \{x_i^n\}$, $x = \{x_i\}$ with $x_{k-1}=1$, $x_k \neq 1$. It follows that $x_i^n = x_i$ for all $i \geqq k$ and all $n$. Because $K$ is compact there is a subsequence $\{x^{n_j}\}_j$ such that $y_i = \lim_{j\to\infty} x_i^{n_j}$ exists for $i \leqq k-1$. Let $l$ be the largest index with $y_l = 1$ ($l=0$ if all $y_i \neq 1$). For $l < i \leqq k-1$ the $y_i$ are isolated points in $K$, hence there exists a $j_0$ with $x_i^{n_j} = y_i$ if $j \geqq j_0$ and $l < i \leqq k-1$, especially $\{x^{n_j}\}_j$ is eventually constant if $l=0$. Now let $y = \{y_i'\}$ with $y_i' = 1$ if $i \leqq l$, $y_i' = y_i$ if $l < i \leqq k-1$ and $y_i' = x_i$ if $i \geqq k$. For $\eta > 0$ there exists $n_0 \geqq j_0$ such that $|1-x_l^{n_j}| \leqq \eta$ for all $j \geqq n_0$. This means $x^{n_j} \in U_\eta(y)$ for all $j \geqq n_0$, hence $\lim_{j\to\infty} x^{n_j} = y \in U_\varepsilon(x)$. This proves that $U_\varepsilon(x)$ is compact.

Now we define a homeomorphism $\tau_n$ of $X$: If $x = \{x_i\} \in X$ let $\tau_n x = \{y_i\}$ be the element with components $y_i = x_i$ if $i \neq n$, $y_n = \tau x_n$, $\tau$ the previously defined homeomorphism of $K$. Evidently $\tau_n$ is one-to-one and onto. If $x_n \neq 1$, then $\tau_n U_\varepsilon(x) = U_\varepsilon(\tau_n x)$. If $x_{n+1} = 1$, then $\tau_n x = x$ and $\tau_n U_\varepsilon(x) = U_\varepsilon(x)$. Finally if $x_n = 1$ but $x_{n+1} \neq 1$, choose $\varepsilon' > 0$ such that $t \in K$, $|1-t| \leqq \varepsilon'$ implies $|1-\tau t| \leqq \varepsilon$. Then $\tau_n U_{\varepsilon'}(x) \subset U_\varepsilon(x)$. It follows that $\tau_n$ is continuous. The same proof works for $\tau_n^{-1}$, hence $\tau_n$ is a homeomorphism of $X$.

Let $G$ be the group of transformations generated by all $\tau_n$. It is easy to see that $G$ is free abelian with the $\tau_n$ as free generators. Considering $G$ as a discrete group, we now have a separable locally compact transformation space $(G, X)$.

Let $F_n$ be the subgroup generated by $\tau_1, \tau_2, \ldots, \tau_{n-1}$. If $x = \{x_i\}$, $x_i = 1$ for $i < k$, $x_k \neq 1$, then $F_k$ is the fixed group of $x$. To get a $G$-space with free action of $G$ take the infinite Tihonov-product of circles $\boldsymbol{T}_i = \boldsymbol{T}$: $Y = \prod_{i=1}^\infty \boldsymbol{T}_i$. Take an element $\zeta$ in $\boldsymbol{T}$ of infinite order, i.e. $\zeta = e^{2\pi i \varphi}$, $\varphi$ irrational, and define the action of $\tau_n \in G$ on $Y$ by

$$\tau_n\{t_i\}_{i\in \boldsymbol{N}} = \{t_i'\}_{i\in \boldsymbol{N}}, \qquad t_i' = t_i \text{ if } i \neq n,\ t_n' = \zeta t_n.$$

It is clear that $Y$ is a compact $G$-space and that the action of $G$ on $Y$ is free: If $\alpha = \prod_1^\infty \tau_n^{\nu_n} \in G$, $t = \{t_i\} \in Y$ and $\alpha t = \{\zeta^{\nu_i} t_i\} = t$, then $\zeta^{\nu_i} = 1$, i.e. $\nu_i = 0$ for all $i \in \boldsymbol{N}$. Finally put

$$Z = X \times Y, \qquad \alpha\{x, y\} = \{\alpha x, \alpha y\}$$

for $\alpha \in G$, $\{x, y\} \in Z$.

THEOREM 1. *$(G, Z)$ is a separable locally compact transformation space without minimal closed invariant nonempty subspaces. $G$ acts freely on $Z$.*

We only have to prove that an invariant closed subset $A \neq \varnothing$ in $Z$ contains an invariant closed nonempty proper subset.

The subset $X_k$ of all $x=\{x_i\}\in X$ with $x_i=1$ for $i<k$ is closed and $G$-invariant, hence $Z_k=X_k\times Y$ is closed and invariant in $Z$, furthermore $\bigcap_{k=1}^{\infty} Z_k=\varnothing$. It follows that for the closed invariant nonempty subset $A$ of $Z$ there exists a least number $n$ with $A\not\subset Z_{n+1}$. Take $a=\{x, y\}\in A$ with $x=\{1,\ldots,1,x_n,x_{n+1},\ldots\}$, $x_n\neq 1$, $y=\{y_i\}$ and choose a sequence $\{n_i\}$ in $N$ such that $\lim_{i\to\infty}\tau^{n_i}y_n=y'_n$ exists. Then we have $\lim \tau^{n_i}x_n=1$ in $K$, hence

$$\lim_{i\to\infty}\tau_n^{n_i}a = a' = \{x', y'\}$$

with $x'=\{1,1,\ldots,1,x_{n+1},\ldots\}$, $y'=\{y'_i\}$, $y'_i=y_i$ if $i\neq n$. Because $A$ is closed and $G$-invariant, $a'$ is in $A$ and also in $Z_{n+1}$. This proves that $A'=A\cap Z_{n+1}$ is a closed invariant nonempty proper subset of $A$. As a corollary of Theorem 1 we have

*$\mathscr{C}^*(G,Z)$ is a separable $C^*$-algebra without maximal closed ideals.*

Unfortunately $\mathscr{C}^*(G,Z)$ is not postliminal, because the orbits of elements $\{x,y\}$ with $x_1=1$ are not discrete, see [9, Théorème 7.7]. On the other hand it is easy to see that all orbits in $X$ are discrete, but $G$ does not act freely on $X$, hence $\alpha$ is no longer a homeomorphism. Anyhow we can prove

THEOREM 2. *$\mathscr{C}^*(G, X)$ is a postliminal separable $C^*$-algebra without maximal closed ideals.*

We will construct explicitly all irreducible unitary representations of $\mathscr{A}^*=\mathscr{C}^*(G, X)$ and apply weak containment arguments to prove that no primitive ideal can be maximal. Of course we could use general theory to get the results but it is instructive and not hard to investigate $\mathscr{A}$ in a naive and direct way. We start with some remarks on $(G, X)$. Let $F_n$ be the subgroup of $G$ generated by $\tau_i$, $1\leqq i<n$, and $G_n$ the complementary subgroup generated by the $\tau_i$ with $i\geqq n$. We have $G=F_n\oplus G_n$ and $G_n=(\tau_n)\oplus G_{n+1}$. For $x=\{x_i\}\in X$ let $n(x)$ be the least index $n$ with $x_n\neq 1$ and let $l(x)$ be equal to the index $l$, for which $x_i=c_l\in K$ for almost all $i$. It is clear that $F_{n(x)}$ is the fixed group of $x$ and that $\alpha\to\alpha x$ is a 1:1-mapping of $G_{n(x)}$ onto the orbit $Gx$ of $x$ in $X$. Let $x^\sim=\mathrm{cl}\,(Gx)$ be the closed hull of the orbit of $x$. We have

$y\in x^\sim$ if and only if $n(y)\geqq n(x)$ and $l(y)=l(x)$, especially
$x^\sim=y^\sim$ if and only if $x$ and $y$ are in the same orbit, i.e.
$y=\alpha x$ with $\alpha\in G$.

**Proof.** Let $y\in x^\sim$. Then clearly $n(y)\geqq n(x)$. Because $l(\alpha x)=l(x)$ for all $\alpha\in G$ and $l(y)=l(z)$ for all $z\in U_\varepsilon(y)$, we also have $l(y)=l(x)$. On the other hand if $n(y)\geqq n(x)$, $l(y)=l(x)$, there is a unique $\beta\in G_{n(y)}$ such that $\beta x$ and $y$ have the same components $y_i$ for $i\geqq n(y)$. It is easy to see that $\lim_{k\to\infty}\tau^k_{n(y)-1}\beta x=y$. If $x^\sim=y^\sim$ then $n(x)=n(y)$ and $l(x)=l(y)$. But these equalities are necessary and sufficient for $Gx=Gy$.

It follows that the space $X^\sim$ of orbit closures $x^\sim$ is in a 1:1-correspondence with $\boldsymbol{N}\times\boldsymbol{Z}$.

Now let $x \in X$, $n = n(x)$, $l = l(x)$ and let $\chi$ be a character of $F_n$. If $\varphi_x$ is the homomorphism $u \to u(x)$ from $\mathscr{C}_\infty(X)$ onto $\boldsymbol{C}$, then the pair $(\chi, \varphi_x)$ defines a covariant representation $\rho$ of $(F_n, X)$ resp. a unitary representation of $\mathscr{L}^1(F_n, \mathscr{C}_\infty(X))$ in $\boldsymbol{C}$: For $u \in \mathscr{L}^1$

$$\rho(u) = \sum_{a \in F_n} \chi(a) u(a)(x).$$

Let P be the representation of $(G, X)$, resp. $\mathscr{L}^1(G, \mathscr{C}_\infty(X))$, resp. $\mathscr{C}^*(G, X)$ induced by $\rho$. $\mathscr{C}_\infty(X)$ is liminal and $F_n$ is the fixed group of $\varphi_x$, hence P is irreducible, see e.g. [7]. Let $\mathfrak{H}$ be the Hilbert space of P, that is the space of functions $\xi \colon G \to \boldsymbol{C}$ with $\xi(ab) = \chi(a)\xi(b)$ for $a \in F_n$ and inner product $(\xi|\eta) = \sum_{g \bmod F_n} (\xi(g)|\eta(g))$. We have a unitary transformation $r$ from $\mathfrak{H}$ onto $\ell^2(G_n)$ defined by

$$(r\xi)(g) = \xi(g), \qquad g \in G_n,$$

the inverse transformation $r^{-1}$ is defined by

$$(r^{-1}\vartheta)(ab) = \chi(a)\vartheta(b), \qquad a \in F_n,\ b \in G_n.$$

Now define a representation

$$\delta = \delta(\chi, x) = (\delta_1, \delta_2)$$

of $(G, X)$ in $\ell^2(G_n)$ by

$$(\delta_1(ab)\xi)(g) = \chi(a)\xi(b^{-1}g), \qquad a \in F_n,\ b, g \in G_n,$$

$$(\delta_2(u)\xi)(g) = u(gx)\xi(g), \qquad u \in \mathscr{C}_\infty(X),\ g \in G_n.$$

We claim that $r\mathrm{P} = \delta r$: For $c = ab \in G$, $a \in F_n$, $b \in G_n$, and $\xi \in \mathfrak{H}$

$$(\mathrm{P}_1(c)\xi)(g) = \xi(c^{-1}g) = \chi(a)\xi(b^{-1}g),$$

hence, if $g \in G_n$,

$$(r\mathrm{P}_1(c)\xi)(g) = \chi(a)\xi(b^{-1}g) = (\delta_1(c)r\xi)(g),$$

similarly $r\mathrm{P}_2(u) = \delta_2(u)r$ for $u \in \mathscr{C}_\infty(X)$. It is easy to see (and of course well known) that the class $\hat{\delta}$ of $\delta$ depends only on the orbit $Gx$ and therefore only on $\chi$, $n = n(x)$ and $l = l(x)$. This justifies the notation $\delta(\chi, n, l)$ for the class of $\delta$ and also for the uniquely defined representation $\delta$ itself.

THEOREM 3. *The mapping* $(\chi, n, l) \to \delta(\chi, n, l)$ *is one-to-one from the disjoint union* $\bigcup_{n=1}^\infty \hat{F}_n \times \boldsymbol{Z}$, $\hat{F}_n$ *the character group of* $F_n$ $(F_1 = (e))$, *onto the dual space* $\hat{\mathscr{C}}$ *of the* $C^*$*-algebra* $\mathscr{C} = \mathscr{C}^*(G, X)$. *The images* $\delta(\mathscr{C})$ *for all* $\delta \in \hat{\mathscr{C}}$ *contain properly the compact operators, so that* $\mathscr{C}$ *is postliminal. Finally* $\ker \delta(\chi, n, l) \subset \ker \delta'(\chi', n', l')$ *if and only if* $\chi = \chi' | F_n$, $n \leqq n'$, *and* $l = l'$.

(For this theorem and the following discussion see e.g. [9, esp. §6].)

Let $\pi = (\pi_1, \pi_2)$ be an irreducible representation of $(G, X)$ in a Hilbert space $\mathfrak{H}$. There exists $x \in X$ such that the hull of $\ker \pi_2$ is equal to the closure $x^\sim$ of the orbit

$Gx$, hence $\pi_2$ may be considered as a representation of $\mathscr{C}_\infty(x^\sim)$, resp. $\pi$ as a covariant representation of $(G, x^\sim)$. It is easy to see that the points $gx$ are open in $x^\sim$ and that the spectral measure $\{P_M\}$, $M$ borel in $x^\sim$, associated with $\pi_2$ is concentrated on $Gx$. It follows that the family of orthogonal projectors $P_g = P_{\{gx\}}$, $g \in G_n$ has the sum $\mathbf{1}$. Let $n = n(x)$, $l = l(x)$. Because $F_n$ acts trivially on $x^\sim$ the $\pi_1(a)$, $a \in F_n$, commute with $\pi_2$, hence with $\pi$. It follows that $\pi_1(a) = \chi(a)\,\mathbf{1}$, where $\chi$ is a character of $F_n$. If $b \in G_n$ then $\pi_1(b)P_{\{gx\}}\pi_1(b)^* = P_{\{b^{-1}gx\}}$, that is $\pi_1(b)P_g\pi_1(b)^* = P_{b^{-1}g}$. Let $E_e$ be a projector with $0 < E_e \leqq P_e$ and let $E_g = \pi_1(g)E_e\pi_1(g)^*$, $g \in G_n$. Then $0 < E_g \leqq P_g$, $E = \sum_{G_n} E_g$ commutes with $\pi_1$ and because $\pi_2(u)P_g = u(gx)P_g$ also with $\pi_2$. This implies $E = \mathbf{1}$, $E_g = P_g$, $\dim P_g = 1$ and $\pi_1(b)P_g\mathfrak{H} = P_{b^{-1}g}\mathfrak{H}$, $b, g \in G_n$. Now we can choose a basis $\{\varepsilon_g\}_{g \in G_n}$ for $\mathfrak{H}$ such that $(\varepsilon_e | \pi_1(g)\varepsilon_g) = 1$ for all $g \in G_n$. This means $\pi_1(g)\varepsilon_e = \varepsilon_{g^{-1}}$ and $\pi_1(a)\varepsilon_g = \pi_1(ag^{-1})\varepsilon_e = \varepsilon_{a^{-1}g}$ for $a, g \in G_n$. For $u \in \mathscr{C}_\infty(X)$ we have $\pi_2(u)\varepsilon_g = \pi_2(u)P_g\varepsilon_g = u(gx)\varepsilon_g$. Now if $\{\varepsilon'_g\}$ is the standard basis of $\ell^2(G_n)$ it is clear that $U\colon \varepsilon'_g \to \varepsilon_g$ is an isomorphism of $\ell^2(G_n)$ carrying $\delta(\chi, n, l)$ into $\pi$. Hence $\pi$ and $\delta(\chi, n, l)$ are equivalent. It is obvious that $\delta(\chi, n, l)$ and $\delta(\chi', n', l')$ are equivalent if and only if $(\chi, n, l) = (\chi', n', l')$.

We may consider $\mathscr{C}_\infty(x^\sim)$ as a subalgebra of $\mathscr{L}^1(G, \mathscr{C}_\infty(x^\sim))$ and $\mathscr{C}^*(G, x^\sim)$. From this it follows that the image of $\mathscr{C}$ under $\delta = \delta(\chi, n, l)$ contains compact and hence all compact operators of $\ell^2(G_n)$. On the other hand if $u \in \mathscr{C}_\infty(x^\sim)$ and $u(y) \neq 0$ for some $y \in x^\sim$, $y \notin Gx$, then $u(gx)$ is not uniformly small outside finite sets in $Gx$ and therefore $\delta_2(u)$ is not compact. This proves that $\mathscr{C}$ is postliminal without liminal factor algebras.

Let $\delta = \delta(\chi, n, l) = (\delta_1, \delta_2)$ and $\delta' = \delta(\chi', n', l') = (\delta'_1, \delta'_2)$. If $\ker \delta \subset \ker \delta'$, in other words if $\delta'$ is weakly contained in $\delta$, then also $\ker \delta_2 \subset \ker \delta'_2$ and $x'^\sim \subset x^\sim$ with $x'^\sim$ and $x^\sim$ the corresponding orbit closures. It follows that $x' \in x^\sim$ and $n' = n(x') \geqq n = n(x)$, $l' = l$. Take $a \in F_n$, $u \in \mathscr{C}_\infty(X)$ with $\delta'_2(u) \neq 0$ and define $f \in \mathscr{L}^1(G, \mathscr{C}_\infty(X))$ by $f(e) = \chi(a)u$, $f(a) = -u$, $f(b) = 0$ for all other $b$ in $G$. Then

$$\delta(f) = \delta_2(\chi(a)u) - \delta_1(a)\delta_2(u) = \chi(a)\delta_2(u) - \chi(a)\delta_2(u) = 0,$$

i.e. $f \in \ker \delta$. It follows that

$$\delta'(f) = \chi(a)\delta'_2(u) - \delta'_1(a)\delta'_2(u) = (\chi(a) - \chi'(a))\delta'_2(u) = 0,$$

hence $\chi(a) = \chi'(a)$ and so $\chi = \chi' | F_n$.

Now let us prove that $\ker \delta(\chi, n, l) \subset \ker \delta(\chi', n', l)$ if $n' > n$ and $\chi = \chi' | F_n$. We take $x$ and $y$ in $X$ with $n(x) = n$, $l(x) = l$, $n(y) = n'$, $l(y) = l$. Then $y^\sim \subset x^\sim$. First we define a homomorphism

$$\omega\colon \mathscr{L}^1(G, \mathscr{C}_\infty(X)) \to \mathscr{L}^1(G_n, \mathscr{C}_\infty(x^\sim))$$

by

$$\omega(u)(g) = \sum_{a \in F_n} \chi(a)u(ag) | x^\sim.$$

Here $w|x^\sim$ for $w \in \mathscr{C}_\infty(X)$ is the restriction of $w$ on $x^\sim$. We have

$$|\omega(u)|_1 = \sum_{G_n} |\omega(u)(g)| \leqq \sum_{a\in F_n} \sum_{g\in G_n} |\chi(a)u(ag)| = |u|_1,$$

hence $\omega$ is bounded and linear. For $w \in \mathscr{C}_\infty(X)$, $g \in G$ let $(gw)(x) = w(g^{-1}x)$. Then the convolution of $u$ and $v$ from $\mathscr{L}^1(G, \mathscr{C}_\infty(X))$ is given by

$$u * v(s) = \sum_{t\in G} tu(st)v(t^{-1}).$$

Put $w^\sim = w|x^\sim$ for $w \in \mathscr{C}_\infty(X)$. Because $F_n$ acts trivially on $x^\sim$ and $\mathscr{C}_\infty(x^\sim)$ we have

$$\omega(u * v)(g) = \sum_{a,\, b\in F_n} \sum_{t\in G_n} \chi(a)tu(agbt)^\sim v(b^{-1}t^{-1})^\sim.$$

The substitution $a \to ab$, $b \to b^{-1}$ yields

$$\begin{aligned}\omega(u * v)(g) &= \sum_{a,b,t} \chi(a)tu(agt)^\sim \chi(b)v(bt^{-1})^\sim \\ &= \sum_{t\in G_n} t\Big(\sum_a \chi(a)u(agt)^\sim\Big)\Big(\sum_b \chi(b)v(bt^{-1})^\sim\Big) \\ &= \sum_{G_n} t\omega(u)(gt)\omega(v)(t^{-1}) = (\omega(u) * \omega(v))(g).\end{aligned}$$

Hence $\omega$ is multiplicative. Similarly $\omega(u^*) = \omega(u)^*$. Now define

$$\gamma_1 = \delta_1|G_n, \quad \gamma_1' = \delta_1'|G_n, \quad \gamma_2(u^\sim) = \delta_2(u), \quad \gamma_2'(u^\sim) = \delta_2'(u).$$

Then $\gamma = (\gamma_1, \gamma_2)$ and $\gamma' = (\gamma_1', \gamma_2')$ are well defined representations of $(G_n, \mathscr{C}_\infty(x^\sim))$, resp. $\mathscr{L}^1(G_n, \mathscr{C}_\infty(x^\sim))$ in $\ell^2(G_n)$ and $\ell^2(G_{n'})$ with $\gamma'(a) = \chi'(a)\mathbf{1}$ for $a \in F_{n'} \cap G_n$. For $u \in \mathscr{L}^1(G, \mathscr{C}_\infty(X))$ we have

$$\begin{aligned}\gamma(\omega(u)) &= \sum_{G_n} \delta_1(g)\delta_2\Big(\sum_{F_n} \chi(a)u(ag)\Big) \\ &= \sum_{G_n}\sum_{F_n} \chi(a)\delta_1(g)\delta_2(u(ag)) \\ &= \sum_{c\in G} \delta_1(c)\delta_2(u(c)) = \delta(u)\end{aligned}$$

and similarly $\gamma'(\omega(u)) = \delta'(u)$. Hence $\delta = \gamma \circ \omega$ and $\delta' = \gamma' \circ \omega$ and it suffices to prove $\ker \gamma \subset \ker \gamma'$, where $\gamma$ and $\gamma'$ are considered as representations of $\mathscr{C}^*(G_n, \mathscr{C}_\infty(x^\sim))$.

Let $\{\varepsilon_h\}_{h\in G_n}$ be the standard basis of $\ell^2(G_n)$. For $u \in \mathscr{L}^1(G_n, \mathscr{C}_\infty(x^\sim))$ we have

$$\gamma(u)\varepsilon_h = \sum_{G_n} \delta_1(g)\delta_2(u(g))\varepsilon_h = \sum_{G_n} u(g)(hx)\varepsilon_{g^{-1}h}$$

hence $\gamma(u) = 0$ implies $u(g)(hx) = 0$ for all $g$ and $h$ in $G_n$. Thus $u(g)$ vanishes on $x^\sim$ for all $g \in G_n$, which means $u = 0$: $\gamma$ is faithful on $\mathscr{L}^1(G_n, \mathscr{C}_\infty(x^\sim))$. But $\gamma$ is the representation induced by the representation $\varphi_x$: $w \to w(x)$ of $\mathscr{C}_\infty(x^\sim)$. It follows from [9, Théorème 4.11] that $\gamma$ is also faithful on $\mathscr{C}^*(G_n, \mathscr{C}_\infty(x^\sim))$, hence $\ker \gamma = 0 \subset \ker \gamma'$ and Theorem 3 is proved.

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